(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2017/2018 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE (MATHEMATICS)

COURSE CODE: STA 443

COURSE TITLE: PROBABILITY AND MEASURE

DATE: 20/12/17 **TIME:** 8 AM -10 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

ANSWER QUESTION ONE AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

1. (a) i. Explain the following terms: an algebra and a sigma algebra (2 mks)

   ii. Let $E_1, E_2 \in \mathcal{F}_0$. Prove that $\mathcal{F}_0$ is an algebra and not a σ-algebra (4 mks)

   (b) A random variable X has a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$. Find the distribution function of X (3 mks)

   (c) If μ is a σ-finite measure on an algebra A of subsets of S. Show that:

   i. there exists an increasing sequence and (4 mks)

   ii. there exists a disjoint σ-finite sequence. (4 mks)

   (d) If $A \subset B$, show that $\mu^*(A) \leq \mu^*(B)$. (3 mks)

   (e) Suppose X is a random variable with distribution μ_X, and g is a Borel measurable function. Show that

$$E[g(X)] = \int_R g(x) d\mu_X$$

(5 mks)

   (f) State and briefly explain any two types of measures on the intervals over the real line. (5 mks)

QUESTION TWO (20 MARKS)

2. (a) Let $\{E_i \subset R^n : i \in N\}$ is countable collection of R^n. Show that

$$\mu^*(\cup_i^\infty E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

(5 mks)

(b) Let $f_{XY}(x, y) = \frac{1}{50}(x^2 + y^2)$ if $0 < x < 2, 1 < y < 4$ and zero otherwise. Find $P(X + Y > 4)$ (5 mks)

(c) Suppose A and B are independent events in the sample space. Show that A^c and B are independent. (5 mks)

(d) Prove that every monotone function is measurable. (5 mks)

QUESTION THREE (20 MARKS)

3. (a) Find the integral $f(x, y) = x^2 + y^2$, on the domain

$$D = \left\{(x, y) \in R^2 : 0 < x < 1, x^2 < y < x\right\}$$

(8 mks)

(b) Suppose $f = \sum_i x_i I_{Ai}$ is a non-negative simple function, $\{A_i\}$ being decomposition of S into F sets, show that

$$\int f d\mu = \sum_i x_i \mu(A_i)$$

(6 mks)

(c) Let $r, s, t \in [1, \infty]$ satisfy $\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$. Prove that for all measurable f and g defined on a space (X, A, μ), given $||fg||_t \leq ||f||_r ||g||_s$ (6 mks)

QUESTION FOUR (20 MARKS)

4. (a) State and explain two properties of conditional expectation (4 mks)

   (b) Find the mathematical expectation of a random variable with:

      i. uniform distribution over the interval $[a, b]$

      ii. triangle distribution

      iii. exponential distribution (6 mks)

   (c) Show that if $\{f_n\}$ is a sequence of non-negative measurable functions, and $\{f_n(x) : n \leq 1\}$ increases monotonically to $f(x)$ for each x then

$$\lim_{n \to \infty} \int_E f_n(x)dm = \int_E f dm$$

(5 mks)

   (d) Suppose $X_1, X_2, \ldots, X_n$ are random variables with finite variance. If $X_1, \ldots, X_n$ are pairwise orthogonal. Show that

$$Var[X_1 + X_2 + ... + X_n] = Var[X_1] + ... + Var[X_n]$$

(6 mks)

QUESTION FIVE (20 MARKS)

5. (a) State Fubini's theorem (2 mks)

   (b) Let f_1 and f_2 be measurable functions on a common domain. Show that each set $\{\omega : f_1(\omega) < f_2(\omega)\}$, $\{\omega : f_1(\omega) = f_2(\omega)\}$ and $\{\omega : f_1(\omega) > f_2(\omega)\}$ is measurable (8 mks)

   (c) Suppose $\{B_n\}$ is sequence of independent events and $\sum_n Pr\{B_n\} = \infty$. Show the probability that B_n occurs infinitely often is one. (10 mks)