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(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2016/2017 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

SPECIAL/ SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

MATHEMATICS

COURSE CODE: MAT 404

COURSE TITLE: DIFFERENTIAL TOPOLOGY

DATE: 21/09/17

TIME: 3 PM -5 PM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of 3 Printed Pages. Please Turn Over.

QUESTION 1 (30 MARKS)

- a) Define a manifold (2 marks)
- b) Show that $X \times Y \subset \mathbb{R}^M \times \mathbb{R}^N$ is a smooth manifold (8 marks)
- c) Let $Z = f^{-1}(y)$ for a regular value y of the mapping $f: X \rightarrow Y$. Prove that
$$\text{Ker}[df_x: T_x X \rightarrow T_y Y] = T_x Z$$
 at any point $x \in Z$. (8 marks)
- d) Prove that a subspace $M \subset \mathbb{R}^k$ is a smooth manifold of dimension n if every point in M has an open neighborhood in M which is diffeomorphic to an open subset of \mathbb{R}^n . (6 marks)
- e) Show that if $\dim(M) = \dim(N)$ and $y \in N$ is a regular value of f then $f^{-1}(y)$ is a discrete set. Furthermore if M is compact then $f^{-1}(y)$ is a finite set. (6 marks)

QUESTION 2 (20 MARKS)

- a) Show that the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a one-dimensional manifold. (8 marks)
- b) Prove that if M is an m -dimensional manifold then the tangent space $T_x(M)$ is an m -dimensional linear space. (6 marks)
- c) State the Inverse function theorem without prove. (2 marks)
- d) Suppose that $f: X \rightarrow Y$ is a diffeomorphism. Prove that $df_x: T_x(X) \rightarrow T_{f(x)}(Y)$ is a linear isomorphism. (4 marks)

QUESTION 3 (20 MARKS)

- a) Prove that an embedding $f: X \rightarrow Y$ maps X diffeomorphically into a submanifold of Y . (10 marks)
- b) Show that every k -dimensional manifold admits a one-to-one immersion in \mathbb{R}^{2k+1} (6 marks)
- c) Show that on any manifold X there exists a proper map $\rho: X \rightarrow \mathbb{R}$ (4 marks)

QUESTION 4 (20 MARKS)

- a) Show that the tangent space does not depend on the choice of parametrization. (6 marks)
- b) Prove that the orthogonal group $O(n)$ is a Lie group (8 marks)
- c) Suppose that M is a compact manifold with boundary and prove that there does not exist a smooth map $f: M \rightarrow \partial M$ that leaves every point of the boundary fixed. (6 marks)

QUESTION 5 (20 MARKS)

- a) Show that a smooth map from the disk D^n to itself has a fixed point. (8 marks)
- b) Let $f: M \rightarrow N$ be an imbedding, where M is a (non-empty) compact n -dimensional smooth manifold and N is a connected n -dimensional smooth manifold. Prove that f is a diffeomorphism. (4 marks)
- c) State local immersion theorem without prove. (2 marks)
- d) Suppose that $f: X \rightarrow Y$ is a submersion at x , and $y = f(x)$. Prove that there exists coordinate around x and y such that $f(x_1, \dots, x_k) = [x_1, \dots, x_l]$ on neighbourhoods $N(x)$ and $M(y)$. (6 marks)