(Knowledge for Development)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2017/2018 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

SPECIAL/ SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

MATHEMATICS

COURSE CODE: **MAT 404**

COURSE TITLE: **DIFFERENTIAL TOPOLOGY**

DATE: 11/10/18 **TIME:** **3 PM -5 PM**

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of **3 Printed Pages**. Please Turn Over.

QUESTION 1 (30 MARKS)

a) Define a manifold (2 marks)

b) Show that $X \times Y \subset \mathbb{R}^M \times \mathbb{R}^N$ is a smooth manifold (8 marks)

c) Let $Z = f^{-1}(y)$ for a regular value y of the mapping $f : X \to Y$. Prove that

$$Ker[\,df_x : T_x X \to T_y Y] = T_x Z \text{ at any point } x \in Z.$$ (8 marks)

d) Prove that a subspace $\mathrm{M} \subset \mathbb{R}^{\mathrm{k}}$ is a smooth manifold of dimension n if every point in M has an open neighborhood in M which is diffeomorphic to an open subset of $\mathbb{R}^{\mathrm{n}}$. (6 marks)

e) Show that if $\dim(\mathrm{M}) = \dim(\mathrm{N})$ and $y \in N$ is a regular value of f then $f^{-1}(y)$ is a discrete set. Furthermore if M is compact then $f^{-1}(y)$ is a finite set. (6 marks)

QUESTION 2 (20 MARKS)

a) Show that the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a one-dimensional manifold. (8 marks)

b) Prove that if M is an $m-$dimensional manifold then the tangent space $T_x(M)$ is an $m-$dimensional linear space. (6 marks)

c) State the Inverse function theorem without prove. (2 marks)

d) Suppose that $f : X \to Y$ is a diffeomorphism. Prove that $df_x : T_x(X) \to T_{f(x)}(Y)$ is a linear isomorphism. (4 marks)

QUESTION 3 (20 MARKS)

a) Prove that an embedding $f : X \to Y$ maps X diffeomorphically into a submanifold of Y. (10 marks)

b) Show that every $k-$dimensional manifolds admits a one-to-one immersion in $\mathbb{R}^{2k+1}$ (6 marks)

c) Show that on any manifold X there exists a proper map $\rho : X \to \mathbb{R}$ (4 marks)

QUESTION 4 (20 MARKS)

a) Show that the tangent space does not depend on the choice of parametrization. (6 marks)

b) Prove that the orthogonal group O(n) is a Lie group (8 marks)

c) Suppose that M is a compact manifold with boundary and prove that there does not exist a smooth map $f: M \to \partial M$ that leaves every point of the boundary fixed. (6 marks)

QUESTION 5 (20 MARKS)

a) Show that a smooth map from the disk D^n to itself has a fixed point. (8 marks)

b) Let $f: M \to N$ be an imbedding, where M is a (non-empty) compact $n-$dimensional smooth manifold and N is a connected $n-$dimensional smooth manifold. Prove that f is a diffeomorphism. (4 marks)

c) State local immersion theorem without prove. (2 marks)

d) Suppose that $f: X \to Y$ is a submersion at x, and $y = f(x)$. Prove that there exists coordinate around x and y such that $f(x_1, ..., x_k) = [x_1, ..., x_l]$ on neighbourhoods $N(x)$ and $M(y)$. (6 marks)