(*Knowledge for Development*)

KIBABII UNIVERSITY

UNIVERSITY EXAMINATIONS

2017/2018 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER

SPECIAL/ SUPPLEMENTARY EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

MATHEMATICS

COURSE CODE: **MAT 301**

COURSE TITLE: REAL ANALYSIS II

DATE: 08/10/18 **TIME:** 8 AM -10 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One and Any other TWO Questions

TIME: 2 Hours

This Paper Consists of **4 Printed Pages**. Please Turn Over.

QUESTION ONE (30 MARKS)

a) State and prove the Cauchy-Schwarz inequality. (6 marks)

b) Show that a subset A of a metric space (X, ρ) is open if and only if $A = A^{\circ}$ (where A° is the interior of A). (3 marks)

c) Let (X, ρ) be a metric space. Show that for any subset E of X, the interior of E (i.e. E°) is an open set. (4 marks)

d) Show that any finite subset E of a metric space (X, ρ) is closed. (4 marks)

e) In (ℝ, d), let $E = (a, b] \cup \{c\}$. Find the derived set of E i.e. the set of all limit points of E. (3 marks)

f) Consider (ℝ, d) and let $x_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$.Show that (x_n)converges in (ℝ, d). (3 marks)

g) Let $(X, \rho), (Y, \sigma)$ be metric spaces and E be a non-void subset of X. Let $f: E \to Y$ be a function and p be an isolated point of E, show that f is continuous at p. (3 marks)

h) Let (X, ρ) be a metric space and E be a subset of X. Show that $\overline{E}$ (i. e. the closure of E)is always closed. (4 marks)

QUESTION TWO (20 MARKS)

a) Let (X, ρ) be a metric space and Ebe a subset of X. Show that a point $p \in X$ is a limit point of Eif and only if every open neighbourhood $N(p)$ of $p$contains infinitely many distinct points of E. (6 marks)

b) Consider the set $\mathbb{R}^n$ of all ordered n-tuples $(a_1, a_2, \cdots, a_n)$ where $a_i \in \mathbb{R} \; \forall \, i = 1, 2, \cdots, n$. If $\vec{x} = (x_1, x_2, \cdots, x_n)$and $\vec{y} = (y_1, y_2, \cdots, y_n)$. Define a number $\rho(\vec{x}, \vec{y})$ by

$$\rho(\vec{x}, \vec{y}) = Sup\{|x_1 - y_1|, |x_2 - y_2|, \cdots\cdots, |x_n - y_n|\}$$
$$= Max\{|x_1 - y_1|, |x_2 - y_2|, \cdots\cdots, |x_n - y_n|\}.$$

Show that the function $\rho$is a metric i.e. ($\mathbb{R}^n$, ρ) is a metric space. (7 marks)

c) Define the terms
   i. discrete metric space
   ii. openneighbourhood. (2 marks)

d) Let (X, ρ) be a metric space and $p \in X$. Let ε be a positive real number. Show that the closed neighbourhood $\overline{N}(p; \varepsilon)$ is a closed subset of (X, ρ) (5 marks)

QUESTION THREE (20 MARKS)

a) Let (X,ρ) be a metric space. Show that a finite subset E of X has no limit points i.e. $E^d = \phi$. (3 marks)

b) Define the terms metric and metric space. Let (X,ρ) be a metric space. Let Y be a non-void subsetof X. Define a function σ on $Y \times Y$ by $\sigma(x, y) = \rho(x, y) \quad \forall\, x, y \in Y$ i.e. $\sigma = \rho|_{Y \times Y}$. Show that (Y,σ) is also a metric space. (5 marks)

c) If p, q are any distinct points in a metric space (X,ρ). Show that there are open neighbourhoods $N(p), N(q)$ of p, q respectively such that $N(p) \cap N(q) = \phi$ i.e. $N(p)$ and $N(q)$ are disjoint. (5 marks)

d) Let E be a non-void subset of $\mathbb{R}$ which is bounded above. Consider the Euclidean metric in $\mathbb{R}$. Show that $Sup\, E \in \overline{E}$. Similarly if E is a non-void subset of $\mathbb{R}$ which is bounded below, show that $Inf\, E \in \overline{E}$. (5 marks)

e) Let (X,ρ) be a metric space and E a non-void subset of X. Let p be any given point in X. Explain the meaning of the following
   i. E is bounded in (X,ρ).
   ii. Diameter of E. (2 marks)

QUESTION FOUR (20 MARKS)

a) Let (X,ρ) be a metric space and Y a non-void subset of X. Let $\rho_Y = \rho|_{Y \times Y}$ and $E \subseteq Y$. Show that "E is closed in (Y,ρ_Y)" if and only if there is a subset F of X which is closed in (X,ρ) and such that $E = F \cap Y$. (5 marks)

b) Let (X,ρ) be a metric space and E be a non-void subset of X. Show that E is bounded if and only if diameter of E is finite. (4 marks)

c) Let (X,ρ) be a metric space and (x_n) be a sequence in X. Show that
   i. Show that every convergent sequence is Cauchy.
   ii. Every Cauchy sequence in a metric space is bounded. (7 marks)

d) Let (X,ρ) be a metric space and E a non-void subset of X. Show that $p \in X$ is a limit point of E if and only if there is a sequence (x_n) of distinct points of E (all distinct from p) such that (x_n) converges to p in (X,ρ).

(4 marks)

QUESTION FIVE (20 MARKS)

a) Let (X,ρ) be a metric space and E a non-void subset of X. Let p be any given point in X. Show that $\rho(p,E)=0$ if and only if $p \in \overline{E}$. (5 marks)

b)

c) Give an example of a metric space in which a subset is closed and bounded but not compact. (4 marks)

d) Let (X,ρ) be a metric space and E be a compact subset of X. show that E is closed i.e. compactness implies closedness. (6 marks)

e) Let (X,ρ) and (Y,σ) be metric spaces and $f: X \to Y$ be a function. Show that f is continuous with respect to ρ and σ if and only if for every subset V of Y which is open in (Y,σ) the set $f^{-1}(V)$ is open in (X,ρ). (5 marks)